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Solvable Lie algebras with naturally graded nilradicals and their invariants

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Abstract

The indecomposable solvable Lie algebras with graded nilradical of maximal nilindex and a Heisenberg subalgebra of codimension one are analysed, and their generalized Casimir invariants are calculated. It is shown that rank one solvable algebras have a contact form, which implies the existence of an associated dynamical system. Moreover, due to the structure of the quadratic Casimir operator of the nilradical, these algebras contain a maximal non-abelian quasi-classical Lie algebra of dimension $2n - 1$, indicating that gauge theories (with ghosts) are possible on these subalgebras.

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1. Introduction

Lie algebras and their invariants play a relevant role in physical models, such as the multidimensional cosmological models, the standard model, nuclear collective motions and rotational states in particle and nuclear physics, the Petrov classification in general relativity, quantum mechanics or, more recently, string theory, where isometry groups of high-dimensional spaces are needed [1–3]. Lie algebras also appear as symmetries of dynamical systems, and are therefore deeply related to the conservation laws of physics [4]. Although semisimple Lie algebras occupy a central position within the Lie algebras appearing in physical models (Lorentz algebra, $\mathfrak{su}(N)$ and $\mathfrak{so}(p, q)$ series, symplectic algebras $\mathfrak{sp}(N, \mathbb{R})$, etc.), as well as various semidirect products, like the Poincaré or the inhomogeneous Lie algebras, the class of solvable algebras has shown to be of considerable interest, as follows from their applicability to the theory of completely integrable Hamiltonian systems or non-abelian gauge theories [5, 6]. While the classification of semisimple Lie algebras constitutes a classical result, solvable Lie algebras over the real field \mathbb{R} have been classified only up to dimension six ([7–9] and references therein). The absence of global structural properties for this class, as well as the existence of parametrized families, indicate that a global classification is not

feasible. However, proceeding by structure, important classes have been classified and their invariants determined, like solvable Lie algebras with Heisenberg or triangular nilradical [10, 11], or specific types of solvable Lie algebras, called rigid ones [12].

In [13] the authors analysed solvable Lie algebras having as nilradical a nilpotent Lie algebra $\mathfrak{n}_{n,1}$ of maximal nilindex and an abelian ideal of codimension one. These algebras are interesting in the sense that the nilradical is naturally graded and has the maximal possible nilindex for a nilpotent Lie algebra. Indeed any nilpotent Lie algebra of dimension n and nilindex $n - 1$ can be shown to be a deformation of $\mathfrak{n}_{n,1}$. Moreover, it is the only naturally graded algebra in odd dimension with these properties. In even dimension there exists another naturally graded algebra with maximal nilindex, called Q_{2n} , which contains a maximal Heisenberg subalgebra of codimension one. The purpose of this work is to analyse the real solvable Lie algebras having the latter nilradical, thus completing the study of invariants of solvable algebras having naturally graded nilradicals of maximal nilindex. While various properties like the structure of the generalized Casimir invariants are similar to the case studied in [13], the algebras of this paper present some particularities that make them worthy to be analysed in detail, and that correspond to properties they are lost by contractions. Among these properties there is the constant number of invariants for any dimension, as well as the existence of linear contact forms, which allow the construction of a dynamical system on the corresponding groups and find applications in classical Hamiltonian structures. We also show that the Q_{2n} algebras contain non-abelian quasi-classical nilpotent Lie algebras of codimension one, which are of certain interest for the study of solutions of the Yang–Baxter equations and in gauge theories based on solvable algebras [14].

We apply the Einstein convention and usual notations for tensor algebra. By indecomposable Lie algebras we mean algebras that do not split into a direct sum of ideals. Unless otherwise stated, any Lie algebra considered in this work is defined over the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

2. Invariants of Lie algebras

The invariant operators of the coadjoint representation of Lie algebras provide important information on a physical system, like quantum numbers, energy spectra or the existence of invariant forms. Polynomial invariants are traditionally called Casimir invariants, and occur for semisimple and nilpotent Lie algebras. More generally, algebraic Lie algebras always admit invariants that are rational. For non-algebraic Lie algebras, specially for those that are solvable, we find rational or even transcendental invariants. These also find applications in representation theory or in classical integrable Hamiltonian systems [5, 15]. In fact, the algorithm usually applied to compute these invariants [16, 17], based on a system of linear first order partial differential equations, does not exclude the existence of irrational invariants, nor there is any physical reason for the invariants to be polynomials. In analogy with the classical Casimir operators, nonpolynomial invariants are called generalized Casimir invariants.

Let $\{X_1, \dots, X_n\}$ be a basis of \mathfrak{g} and $\{C_{ij}^k\}$ be the structure constants over this basis. We consider the representation of \mathfrak{g} in the space $C^\infty(\mathfrak{g}^*)$ given by

$$\widehat{X}_i = -C_{ij}^k x_k \partial_{x_j}, \quad (1)$$

where $[X_i, X_j] = C_{ij}^k X_k$ ($1 \leq i < j \leq n, 1 \leq k \leq n$). This representation is easily seen to satisfy the brackets $[\widehat{X}_i, \widehat{X}_j] = C_{ij}^k \widehat{X}_k$. The invariants are functions on the generators $F(X_1, \dots, X_n)$ of \mathfrak{g} such that

$$[X_i, F(X_1, \dots, X_n)] = 0, \quad (2)$$

and are found by solving the system of linear first order partial differential equations:

$$\widehat{X}_i F(x_1, \dots, x_n) = -C_{ij}^k x_k \partial_{x_j} F(x_1, \dots, x_n) = 0, 1 \leq i \leq n. \tag{3}$$

and then replacing the variables x_i by the corresponding generator X_i (possibly after symmetrizing). A maximal set of functionally independent solutions of (3) will be called a fundamental set of invariants. The cardinal $\mathcal{N}(\mathfrak{g})$ of such a set can be described in terms of the dimension and a certain matrix associated with the commutator table. More specifically, denote by $A(\mathfrak{g})$ the matrix representing the commutator table of \mathfrak{g} over a given basis, i.e.,

$$A(\mathfrak{g}) = (C_{ij}^k x_k). \tag{4}$$

Such a matrix has necessarily even rank by antisymmetry. Then $\mathcal{N}(\mathfrak{g})$ is given by

$$\mathcal{N}(\mathfrak{g}) = \dim \mathfrak{g} - \text{rank}(C_{ij}^k x_k). \tag{5}$$

This formula was first described by Beltrametti and Blasi [16]. With respect to the number of independent Casimir operators of \mathfrak{g} , formula (5) is merely an upper bound. For high dimensional Lie algebras, it is sometimes convenient to work with the analogue of formula (5) in terms of differential forms. Let $\mathcal{L}(\mathfrak{g}) = \mathbb{R}\{d\omega_i\}_{1 \leq i \leq \dim \mathfrak{g}}$ be the linear subspace of $\bigwedge^2 \mathfrak{g}^*$ generated by the Maurer–Cartan forms $d\omega_i$ of \mathfrak{g} . If $\omega = a^i d\omega_i$ ($a^i \in \mathbb{R}$) is a generic element of $\mathcal{L}(\mathfrak{g})$, there always exists an integer $j_0(\omega) \in \mathbb{N}$ such that

$$\bigwedge_{j_0(\omega)} \omega \neq 0, \quad \bigwedge_{j_0(\omega)+1} \omega \equiv 0. \tag{6}$$

This equation shows that $r(\omega) = 2j_0(\omega)$ is the rank of the 2-form ω . We now define

$$j_0(\mathfrak{g}) = \max\{j_0(\omega) \mid \omega \in \mathcal{L}(\mathfrak{g})\}. \tag{7}$$

The quantity $j_0(\mathfrak{g})$, which depends only on the structure of \mathfrak{g} , constitutes a numerical invariant of \mathfrak{g} [18]. The number of invariants follows from the expression:

$$\mathcal{N}(\mathfrak{g}) = \dim \mathfrak{g} - 2j_0(\mathfrak{g}). \tag{8}$$

3. Naturally graded nilpotent algebras of maximal index

To any Lie algebra \mathfrak{r} we can naturally associate various recursive series of ideals:

$$D^0 \mathfrak{r} = \mathfrak{r} \supset D^1 \mathfrak{r} = [\mathfrak{r}, \mathfrak{r}] \supset \dots \supset D^k \mathfrak{r} = [D^{k-1} \mathfrak{r}, D^{k-1} \mathfrak{r}] \supset \dots \tag{9}$$

$$C^0 \mathfrak{r} = \mathfrak{r} \supset C^1 \mathfrak{r} = [\mathfrak{r}, \mathfrak{r}] \supset \dots \supset C^k \mathfrak{r} = [\mathfrak{r}, D^{k-1} \mathfrak{r}] \supset \dots \tag{10}$$

called respectively the derived and central descending sequence. Solvability is given whenever the derived series is finite, i.e., if there exists a k such that $D^k \mathfrak{r} = 0$, and nilpotency whenever the central descending sequence is finite, i.e., if $C^k \mathfrak{r} = 0$ for some k . The dimensions of the subalgebras in both series provide numerical invariants of the Lie algebra. We use the notation DS and CDS for the dimension sequences of the descending and central descending sequences, respectively.

Starting from the central descending sequence, we can associate a graded Lie algebra to \mathfrak{r} , which is usually denoted by $\mathfrak{gr}(\mathfrak{r})$,

$$\mathfrak{g}_{i+1} := \frac{C^i \mathfrak{r}}{C^{i+1} \mathfrak{r}}, \quad i \geq 0. \tag{11}$$

It satisfies the condition

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \quad 1 \leq i, j. \tag{12}$$

A Lie algebra is said naturally graded if \mathfrak{r} and $\mathfrak{gr}(\mathfrak{r})$ are isomorphic Lie algebras.

Example 1. Consider the six-dimensional nilpotent Lie algebra given by the brackets

$$\begin{aligned} [X_2, X_6] &= [X_3, X_4] = X_1, \\ [X_3, X_5] &= [X_4, X_6] = X_2, \\ [X_4, X_5] &= X_3, [X_5, X_6] = X_4 \end{aligned} \tag{13}$$

over the basis $\{X_1, \dots, X_6\}$. It is isomorphic to the Lie algebra $A_{6,22}$ of the classification in [19], and satisfies $DS = [6, 4, 1, 0]$ and $CDS = [6, 4, 3, 2, 1, 0]$. The associated graded Lie algebra $\text{gr}(A_{6,22})$ has the brackets

$$\begin{aligned} [X_2, X_6] &= [X_3, X_4] = X_1, \\ [X_3, X_5] &= X_2, \\ [X_4, X_5] &= X_3, [X_5, X_6] = X_4. \end{aligned} \tag{14}$$

Therefore $A_{6,22}$ is not naturally graded, since it is not isomorphic to $\text{gr}(A_{6,22})$. However, it can be shown that $\text{gr}(A_{6,22})$ is a Inönü–Wigner contraction of $A_{6,22}$.

Indeed, it can be shown that the graded algebras $\text{gr}(\mathfrak{n})$ are always a contraction of the Lie algebra \mathfrak{n} . Therefore the naturally graded nilpotent Lie algebras play a central role in the classification of nilpotent Lie algebras.

In [13] the solvable Lie algebras having the n -dimensional nilpotent Lie algebras $\mathfrak{n}_{n,1}$ defined by

$$[X_1, X_i] = X_{i+1}, \quad 2 \leq i \leq n - 1 \tag{15}$$

as nilradical were analysed. The $\mathfrak{n}_{n,1}$ algebra has maximal nilindex and also has an abelian subalgebra of maximal dimension, generated by $\{X_2, \dots, X_n\}$. It can be easily verified that this algebra is naturally graded. The question whether there exist other naturally graded nilpotent Lie algebras of dimension n and nilindex $n - 1$ was answered in [20].

Proposition 1. *Let \mathfrak{n} be a naturally graded nilpotent Lie algebra of dimension n and central descending sequence $CDS = [n, n - 2, n - 3, \dots, 2, 1, 0]$. Then \mathfrak{n} is isomorphic to $\mathfrak{n}_{n,1}$ if n is odd, and isomorphic to $\mathfrak{n}_{n,1}$ or Q_{2n} if n is even, where*

(i) $\mathfrak{n}_{n,1}$:

$$[X_1, X_i] = X_{i+1}, \quad 2 \leq i \leq n - 1$$

over the basis $\{X_1, \dots, X_n\}$,

(ii) Q_{2m} ($m \geq 3$):

$$\begin{aligned} [X_1, X_i] &= X_{i+1}, & 2 \leq i \leq 2m - 2 \\ [X_k, X_{2n+1-k}] &= (-1)^k X_{2m}, & 2 \leq k \leq m. \end{aligned}$$

over the basis $\{X_1, \dots, X_{2m}\}$.

We mention that for $n = 2m = 4$ the Lie algebras $\mathfrak{n}_{4,1}$ and Q_4 are isomorphic. In higher dimensions, they are no more isomorphic, but related by a generalized Inönü–Wigner contraction. More precisely, consider the automorphism of Q_{2n} given by the matrix

$$(X'_1, \dots, X'_{2n})^T = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & \varepsilon & 0 & \dots & 0 & 0 \\ 0 & 0 & \varepsilon & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ \vdots \\ X_{2n-1} \\ X_{2n} \end{pmatrix}.$$

Over the new basis the brackets of Q_{2n} are expressed as

$$\begin{aligned} [X'_1, X'_i] &= X'_{i+1}, & 2 \leq i \leq 2n - 1 \\ [X'_k, X'_{2n+1-k}] &= (-1)^k \varepsilon X'_{2n}, & 2 \leq k \leq n. \end{aligned} \tag{16}$$

In the limit $\varepsilon \rightarrow 0$ we obtain the Lie algebra $\mathfrak{n}_{2n,1}$. In spite of this fact, the nilpotent Lie algebras Q_{2n} and $\mathfrak{n}_{n,1}$ have a very different behaviour. While the second has a number of Casimir operators which depends on the dimension [13], the algebra Q_{2n} has a fixed number of invariants for any n .

Proposition 2. *For any $n \geq 3$ the nilpotent Lie algebra Q_{2n} has exactly 2 Casimir operators, given by the symmetrization of the following functions:*

$$I_1 = x_{2n} \tag{17}$$

$$I_2 = x_1 x_{2n} + x_3 x_{2n-1} + \sum_{k=4}^n (-1)^{k+1} x_k x_{2n+2-k} + \frac{(-1)^n}{2} x_{n+1}^2. \tag{18}$$

Proof. The Maurer–Cartan equations of the algebra Q_{2n} are

$$\begin{aligned} d\omega_1 &= d\omega_2 = 0, \\ d\omega_{j+1} &= \omega_1 \wedge \omega_j, & 2 \leq j \leq 2n - 2 \\ d\omega_{2n} &= \sum_{k=2}^n (-1)^k \omega_k \wedge \omega_{2n+1-k}. \end{aligned} \tag{19}$$

The 2-form $\omega = d\omega_{2n}$ is of maximal rank, therefore $j_0(Q_{2n}) = n - 1$ and by formula (8) we have $\mathcal{N}(Q_{2n}) = 2$. Clearly the generator of the centre is one Casimir operator of the algebra. In order to determine the other independent invariant, we have to solve the system (3):

$$\widehat{X}_1 F := \sum_{j=2}^{2n-2} x_{j+1} \frac{\partial F}{\partial x_j} = 0 \tag{20}$$

$$\widehat{X}_j F := -x_{j+1} \frac{\partial F}{\partial x_1} + (-1)^j x_{2n} \frac{\partial F}{\partial x_{2n+1-j}} = 0 \tag{21}$$

$$\widehat{X}_{2n-1} F := -x_{2n} \frac{\partial F}{\partial x_2} = 0, \tag{22}$$

where $2 \leq j \leq 2n - 2$. Equation (22) implies that $\frac{\partial F}{\partial x_2} = 0$. For any fixed $2 \leq j \leq 2n - 2$, the function $x_1 x_{2n} + (-1)^j x_{j+1} x_{2n+1-j}$ is a solution of equation (21). If we consider the function $C = x_1 x_{2n} + x_3 x_{2n-1} + \sum_{k=4}^n (-1)^{k+1} x_k x_{2n+2-k} + \frac{(-1)^n}{2} x_{n+1}^2$, for any $j \geq 3$ the following identity is satisfied:

$$x_{j+2} \frac{\partial C}{\partial x_{j+1}} + x_{2n+1} \frac{\partial C}{\partial x_{2n-j}} = 0. \tag{23}$$

This implies that $\widehat{X}_1(C) = 0$, and therefore that C is an invariant of the algebra.

The Casimir operator follows at once replacing x_i by X_i . Observe in particular that C coincides with its symmetrization. \square

3.1. Solvable Lie algebras with fixed nilradical

Any solvable Lie algebra \mathfrak{r} over the real or complex field admits a decomposition

$$\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n} \tag{24}$$

satisfying the relations

$$[\mathfrak{t}, \mathfrak{n}] \subset \mathfrak{n}, \quad [\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}, \quad [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{n}, \tag{25}$$

where \mathfrak{n} is the maximal nilpotent ideal of \mathfrak{r} , called the nilradical, and $\overrightarrow{\oplus}$ denotes the semidirect product. It was proven in [8] that the dimension of the nilradical satisfies the following inequality:

$$\dim \mathfrak{n} \geq \frac{1}{2} \dim \mathfrak{r}. \tag{26}$$

Applying the Jacobi identity to any elements $X \in \mathfrak{t}, Y_1, Y_2 \in \mathfrak{n}$, we obtain that

$$[X, [Y_1, Y_2]] + [Y_2, [X, Y_1]] + [Y_1, [Y_2, X]] = 0, \tag{27}$$

i.e., $\text{ad}(X)$ acts as a derivation of the nilpotent algebra \mathfrak{n} . Since the elements $X \notin \mathfrak{n}$, these derivations are not nilpotent, and given a basis $\{X_1, \dots, X_n\}$ of \mathfrak{t} and arbitrary scalars $\alpha_1, \dots, \alpha_n \in \mathbb{R} - \{0\}$, we have that

$$(\alpha_1 \text{ad}(X_1) + \dots + \alpha_n \text{ad}(X_n))^k \neq 0, \quad k \geq 1, \tag{28}$$

that is, the matrix $\alpha_1 \text{ad}(X_1) + \dots + \alpha_n \text{ad}(X_n)$ is not nilpotent. We say that the elements X_1, \dots, X_n are nil-independent [8]. This fact imposes a first restriction on the dimension of a solvable Lie algebra having a given nilradical, namely, that $\dim \mathfrak{r}$ is bounded by the maximal number of nil-independent derivations of the nilradical. Therefore, the classification of solvable Lie algebras reduces to the problem of finding all non-equivalent extensions determined by a set of nil-independent derivations. The equivalence of extensions is considered under the transformations of the type

$$X_i \mapsto a_{ij} X_j + b_{ik} Y_k, \quad Y_k \mapsto R_{kl} Y_l, \tag{29}$$

where (a_{ij}) is an invertible $n \times n$ matrix, (b_{ik}) is a $n \times \dim \mathfrak{n}$ matrix and (R_{kl}) is an automorphism of the nilradical \mathfrak{n} .

4. Derivations of Q_{2n}

In this section we determine the algebra of derivations of Q_{2n} in order to obtain all equivalence classes of extensions by non-nilpotent derivations.

Proposition 3. Any outer derivation $f \in \text{Der}(Q_{2n})$ has the form

$$\begin{aligned} f(X_1) &= \lambda_1 X_1 + f_1^{2n} X_{2n} \\ f(X_2) &= \lambda_2 X_2 + \sum_{k=2}^{n-1} f_2^{2k+1} X_{2k+1} \\ f(X_{2+t}) &= (t\lambda_1 + \lambda_2) X_{2+t} + \sum_{k=2}^{\lfloor \frac{2n-3-t}{2} \rfloor} f_2^{2k+1} X_{2k+1+t}, \quad 1 \leq t \leq 2n - 4 \\ f(X_{2n-1}) &= ((2n - 3)\lambda_1 + \lambda_2) X_{2n-1} \\ f(X_{2n}) &= ((2n - 3)\lambda_1 + 2\lambda_2) X_{2n}. \end{aligned} \tag{30}$$

In particular $\dim \text{Der}(Q_{2n}) / I \text{Der}(Q_{2n}) = n + 1$.

Proof. For convenience we denote for any $1 \leq i \leq 2n$

$$f(X_i) = \sum_{j=1}^{2n} f_i^j X_j, \tag{31}$$

where the f_i^j are scalars. Since any derivation maps central elements onto central elements, we have that

$$f(X_{2n}) = f_{2n}^{2n} X_{2n}. \tag{32}$$

The condition

$$[f(X_1), X_2] + [X_1, f(X_2)] = f(X_3) \tag{33}$$

shows that

$$f(X_3) = (f_1^1 + f_2^2)X_3 + \sum_{k=3}^{2n-2} f_2^k X_{k+1} - f_1^{2n} X_{2n}. \tag{34}$$

Since $X_{2+t} = \text{ad}(X_1)^t(X_2)$ for $1 \leq t \leq 2n - 3$, iteration of equation (33) shows that

$$f(X_{2+t}) = (tf_1^1 + f_2^2)X_{2+t} + \sum_{k=3}^{2n-1-t} f_2^k X_{k+t} + (-1)^t f_1^{2n-1-t} X_{2n}. \tag{35}$$

The condition

$$[f(X_1), X_{2n-1}] + [X_1, f(X_{2n-1})] = 0$$

implies that $f_1^2 = 0$. In particular it follows that

$$f(X_{2n-1}) = ((2n - 3)f_1^1 + f_2^2)X_{2n}. \tag{36}$$

We now evaluate the Leibniz condition on the pairs (X_2, X_{2+t}) for $1 \leq t \leq 2n - 4$:

$$[f(X_2), X_{2+t}] + [X_2, f(X_{2+t})] = f_2^{2n-1-t}(1 - (-1)^t)X_{2n} + f_2^1 X_{3+t} = 0,$$

from which we deduce that $f_2^1 = 0$ and

$$f_2^1 = 0 \quad f_2^{2n-1-t} = 0, \quad t = 1, 3, \dots, 2n - 5. \tag{37}$$

It can be easily shown that for all $k = 2, \dots, n$ we obtain

$$[f(X_k), X_{2n+1-k}] = ((2n - 3)f_1^1 + 2f_2^2)X_{2n} = f(X_{2n}).$$

The remaining brackets give no new conditions on the coefficients f_i^j . Therefore any derivation f has the form:

$$\begin{aligned} f(X_1) &= f_1^1 X_1 + \sum_{k=3}^{2n} f_1^k X_{2n} \\ f(X_2) &= f_2^2 X_2 + \sum_{k=2}^{n-1} f_2^{2k+1} X_{2k+1} + f_2^{2n} X_{2n} \\ f(X_{2+t}) &= (tf_1^1 + f_2^2)X_{2+t} + \sum_{k=2}^{\lfloor \frac{2n-3-t}{2} \rfloor} f_2^{2k+1} 2k + 1 + t + (-1)^t f_1^{2n-1-t} X_{2n}, \quad 1 \leq t \leq 2n - 4 \\ f(X_{2n-1}) &= ((2n - 3)\lambda_1 + \lambda_2)X_{2n-1} \\ f(X_{2n}) &= ((2n - 3)\lambda_1 + 2\lambda_2)X_{2n}. \end{aligned} \tag{38}$$

Since there are $3n$ parameters, we conclude that

$$\dim \text{Der}(Q_{2n}) = 3n.$$

For any of these parameters we define the following derivations:

$$\begin{aligned} F_1^1(X_1) &= X_1, & F_1^1(X_j) &= (j - 2)X_j, & F_1^1(X_{2n}) &= (2n - 3)X_{2n}, & 3 \leq j \leq 2n - 1 \\ F_2^2(X_2) &= X_2, & F_2^2(X_j) &= X_j, & F_2^2(X_{2n}) &= 2X_{2n}, & 3 \leq j \leq 2n - 1 \\ F_1^k(X_1) &= X_k, & F_1^k(X_{2n+2-k}) &= (-1)^k X_{2n}, & 3 \leq j \leq 2n - 1 \\ F_1^{2n}(X_1) &= X_{2n}, \\ F_1^{2k+1}(X_2) &= X_{2k+1}, & F_2^{2k+1}(X_{2+t}) &= X_{2k+1+t}, & 1 \leq t \leq 2(n - 1 - k), & 2 \leq k \leq n - 1 \\ F_1^{2n}(X_2) &= X_{2n}. \end{aligned} \tag{39}$$

To obtain the outer derivations, we have to determine all adjoint operators $\text{ad}(X)$ for $X \in Q_{2n}$: It can be easily seen that the following relations hold:

$$\text{ad } X_1 = F_2^3, \quad \text{ad } X_2 = F_1^3, \quad \text{ad } X_k = F_1^k (3 \leq k \leq 2n - 2), \quad \text{ad } X_{2n-1} = F_2^{2n}.$$

Therefore there are $n + 1$ outer derivations, corresponding to the derivations $\{F_1^1, F_1^{2n}, F_2^2, F_2^{2k+1}\}_{1 \leq k \leq 2n-1}$. \square

Corollary 1. Any non-nilpotent outer derivation F of Q_{2n} is of the form

$$F = \alpha_1 F_1^1 + \alpha_2 F_2^2 + \sum_k \beta_k F_2^{2k+1} + \beta_n F_1^{2n}, \tag{40}$$

where either $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$.

5. Solvable Lie algebras with Q_{2n} -nilradical

In this section we apply the preceding results on derivations and equation (40) to classify the solvable real and complex Lie algebras having a nilradical isomorphic to the graded algebra Q_{2n} .

Proposition 4. Any solvable Lie algebra \mathfrak{v} with nilradical isomorphic to Q_{2n} has dimension $2n + 1$ or $2n + 2$.

The proof is an immediate consequence of corollary 1.

Proposition 5. Any solvable Lie algebra of dimension $2n + 1$ with nilradical isomorphic to Q_{2n} is isomorphic to one of the following algebras:

(i) $\mathfrak{v}_{2n+1}(\lambda_2)$:

$$\begin{aligned} [X_1, X_k] &= X_{k+1}, & 2 \leq k \leq 2n - 2 \\ [X_k, X_{2n+1-k}] &= (-1)^k X_{2n}, & 2 \leq k \leq n \\ [Y, X_1] &= X_1, \\ [Y, X_k] &= (k - 2 + \lambda_2)X_k, & 2 \leq k \leq 2n - 1 \\ [Y, X_{2n}] &= (2n - 3 + 2\lambda_2)X_{2n}. \end{aligned}$$

(ii) $\mathfrak{v}_{2n+1}(2 - n, \varepsilon)$

$$\begin{aligned} [X_1, X_k] &= X_{k+1}, & 2 \leq k \leq 2n - 2 \\ [X_k, X_{2n+1-k}] &= (-1)^k X_{2n}, & 2 \leq k \leq n \\ [Y, X_1] &= X_1 + \varepsilon X_{2n}, & \varepsilon = -1, 0, 1 \\ [Y, X_k] &= (k - n)X_k, & 2 \leq k \leq 2n - 1 \\ [Y, X_{2n}] &= X_{2n}. \end{aligned}$$

(iii) $\mathfrak{r}_{2n+1}(\lambda_2^5 \cdots \lambda_2^{2n-1})$

$$\begin{aligned} [X_1, X_k] &= X_{k+1}, & 2 \leq k \leq 2n - 2 \\ [X_k, X_{2n+1-k}] &= (-1)^k X_{2n}, & 2 \leq k \leq n \\ [Y, X_{2+t}] &= X_{2+t} + \sum_{k=2}^{\lfloor \frac{2n-3-t}{2} \rfloor} \lambda_2^{2k+1} X_{2k+1+t}, & 0 \leq t \leq 2n - 6 \\ [Y, X_{2n-k}] &= X_{2n-k}, & k = 1, 2, 3 \\ [Y, X_{2n}] &= 2X_{2n}. \end{aligned}$$

Moreover, the first nonvanishing parameter λ_2^{2k+1} can be normalized to 1.

Proof. Let $F = \alpha_1 F_1^1 + \alpha_2 F_2^2 + \sum_k \beta_k F_2^{2k+1} + \beta_n F_1^{2n}$ be a non-nilpotent derivation.

(i) Let $\alpha_1 \neq 0$. A scaling change allows us to suppose that $\alpha_1 = 1$. By a change of the type

$$\begin{aligned} X'_{2+t} &= X_{2+t} + \sum_{k=2}^{\lfloor \frac{2n-3-t}{2} \rfloor} \gamma_k X_{2k+1+t}, & 0 \leq t \leq 2n - 4 \\ X'_i &= X_i, & i = 1, 2n - 1, 2n, \end{aligned} \tag{41}$$

we put to zero first f_2^{2n-1} , then f_2^{2n-3} etc up to f_2^5 . This shows that the extension given by F is equivalent to the extension defined by

$$F' = F_1^1 + \alpha_2 F_2^2 + \beta_n F_1^{2n}. \tag{42}$$

If further $\alpha_2 \neq 2 - n$, then the change of basis

$$X'_1 = X_1 + \frac{f_1^{2n}}{2(n - 2 + \alpha_2)} X_{2n} \tag{43}$$

allows us to suppose $f_1^{2n} = 0$. Therefore the derivation is diagonal and has eigenvalues

$$\Delta = (1, \alpha_2, 1 + \alpha_2, \dots, 2n - 3 + \alpha_2, 2n - 3 + 2\alpha_2) \tag{44}$$

over the ordered basis $\{X_1 \cdots X_{2n}\}$ of Q_{2n} . We obtain the solvable Lie algebras $\mathfrak{r}_{2n+1}(\alpha_2)$. However, if $\alpha_2 = 2 - n$ and $f_1^{2n} \neq 0$, then it cannot be removed. The only possibility is to consider scaling transformations. Over $\mathbb{F} = \mathbb{R}$ this allows us to put f_1^{2n} equal to 1 if $f_1^{2n} > 0$ or $f_1^{2n} = -1$ if $f_1^{2n} < 0$, while over $\mathbb{F} = \mathbb{C}$ we can always normalize f_1^{2n} to 1. This gives the Lie algebras $\mathfrak{r}_{2n+1}(2 - n, \varepsilon)$ ¹. In addition, if $\alpha_2 = 2 - n$ but $f_1^{2n} = 0$, we obtain the Lie algebra $\mathfrak{r}_{2n+1}(2 - n)$.

(ii) Let us suppose now that $\alpha_1 = 0$. By nil-independence we have $\alpha_2 \neq 0$ and by scaling transformation we can suppose that $\alpha_2 = 1$. The change of basis

$$X'_1 = X_1 - \frac{1}{2} f_1^{2n} X_{2n} \tag{45}$$

allows us to put f_1^{2n} to zero. Now the parameters f_2^{2k+1} ($2 \leq k \leq n - 1$) cannot be removed, so that unless all vanish, the derivation F is not diagonal. However, the first non-vanishing parameter f_2^{2k+1} can always be normalized to 1 by a scaling change of basis. This case provides the family of algebras $\mathfrak{r}_{2n+1}(f_2^5 \cdots f_2^{2n-1})$. \square

¹ The Lie algebras $\mathfrak{r}_{2n+1}(2 - n, -1)$ and $\mathfrak{r}_{2n+1}(2 - n, 1)$ being isomorphic over \mathbb{C} .

Finally, if we add the two nil-independent elements, there is only one possibility:

Proposition 6. *For any $n \geq 3$ there is only one solvable Lie algebra \mathfrak{r}_{2n+2} of dimension $2n+2$ having a nilradical isomorphic to Q_{2n} :*

$$\begin{aligned} [X_1, X_k] &= X_{k+1}, & 2 \leq k \leq 2n-2 \\ [X_k, X_{2n+1-k}] &= (-1)^k X_{2n}, & 2 \leq k \leq n \\ [Y_1, X_k] &= kX_k & 1 \leq k \leq 2n-1 \\ [Y_1, X_{2n}] &= (2n+1)X_{2n}, \\ [Y_2, X_k] &= X_k, & 2 \leq k \leq 2n-1 \\ [Y_2, X_{2n}] &= 2X_{2n}. \end{aligned}$$

6. The generalized Casimir invariants

We now consider the solvable Lie algebras obtained in the previous section, and compute their generalized Casimir invariants.

Theorem 1. *The Lie algebras $\mathfrak{r}(\lambda_2)$, $\mathfrak{r}(2-n, \varepsilon)$ and $\mathfrak{r}(\lambda_2^5, \dots, \lambda_2^{2n-1})$ have one invariant for any dimension. They can be chosen as follows:*

(i) $\mathfrak{r}_{2n+1}(\lambda_2)$

$$J = I_2 x_{2n}^{-\alpha}, \quad \alpha = \frac{2n-2+2\lambda_2}{2n-3+2\lambda_2}, \quad (46)$$

(ii) $\mathfrak{r}_{2n+1}(2-n, \varepsilon)$

$$J = \frac{1}{x_{2n}^2} I_2 - \varepsilon \ln(x_{2n}), \quad (47)$$

(iii) $\mathfrak{r}_{2n+1}(\lambda_2^5, \dots, \lambda_2^{2n-1})$

$$J = \frac{I_2}{x_{2n}}, \quad (48)$$

where in all cases

$$I_2 = x_1 x_{2n} + x_3 x_{2n-1} + \sum_{j=4}^n (-1)^j x_j x_{2n+2-j} + \frac{(-1)^{n+1}}{2} x_{n+1}^2. \quad (49)$$

Proof. Using the Maurer–Cartan equations of the solvable Lie algebras above, it is straightforward to verify that in all cases we have $\mathcal{N}(\mathfrak{r}) = 1$. If moreover the derivation F defining the extension of Q_{2n} acts nontrivially on the centre X_{2n} , then clearly the invariants are independent on the variable y associated with the generator Y . To find the invariants of the solvable algebras reduces then to solve the equation $\widehat{Y}F = 0$, taking into account the invariants $I_1 = x_{2n}$ and

$$I_2 = x_1 x_{2n} + x_3 x_{2n-1} + \sum_{j=4}^n (-1)^j x_j x_{2n+2-j} + \frac{(-1)^{n+1}}{2} x_{n+1}^2 \quad (50)$$

obtained from proposition 2.

(i) $\mathfrak{r}(\lambda_2)$.

The equation to be solved is

$$\widehat{Y}F := x_1 \frac{\partial F}{\partial x_1} + \sum_{k=2}^{2n-1} (k-2+\lambda_2)x_k \frac{\partial F}{\partial x_k} + (2n-3+2\lambda_2)x_{2n} \frac{\partial F}{\partial x_{2n}} = 0. \quad (51)$$

It can be easily verified that

$$\widehat{Y}(I_1) = (2n-3+2\lambda_2)I_1 \quad \widehat{Y}(I_2) = (2n-2+2\lambda_2)I_2.$$

This means that the Casimir operators of the nilradical are semi-invariants of the solvable extension. This fact always holds for diagonal derivations [10, 12, 13, 21]. Observe that if $2n-3+2\lambda_2 = 0$, then $J = x_{2n}$ is already the invariant of the algebra, while for $2n-2+2\lambda_2 = 0$ the function I_2 is a Casimir operator of $\mathfrak{r}_{2n+1}(2-n)$. If neither of I_1 or I_2 is a solution of (51), then, considering I_1 and I_2 as new variables u and v , we take the differential equation

$$\frac{\partial \Phi}{\partial u} + \frac{(2n-2+2\lambda_2)v}{(2n-3+2\lambda_2)u} \frac{\partial \Phi}{\partial v} = 0, \quad (52)$$

with the general solution

$$\Phi \left(\frac{u^{2n-2+2\lambda_2}}{v^{2n-3+2\lambda_2}} \right). \quad (53)$$

Therefore the invariant of $\mathfrak{r}_{2n+1}(\lambda_2)$ can be taken as

$$J = I_2 x_{2n}^{-\alpha}, \quad \alpha = \left(\frac{2n-2+2\lambda_2}{2n-3+2\lambda_2} \right). \quad (54)$$

(ii) $\mathfrak{r}_{2n+1}(2-n, \epsilon)$.

In this case we have

$$\widehat{Y}F := (x_1 + \epsilon x_{2n}) \frac{\partial F}{\partial x_1} + \sum_{k=2}^{2n-1} (k-n)x_k \frac{\partial F}{\partial x_k} + x_{2n} \frac{\partial F}{\partial x_{2n}} = 0. \quad (55)$$

Since the action is not diagonal when $\epsilon \neq 0$, $\widehat{Y}(I_2)$ will not be a multiple of I_2 . However, replacing I_2 by $I_2 x_{2n}^{-2}$, we obtain

$$\widehat{Y}(I_2 x_{2n}^{-2}) = \epsilon.$$

Since $\widehat{Y}(\ln(x_{2n})) = 1$, adding the logarithm $-\epsilon \ln(x_{2n})$, the function

$$I_2 x_{2n}^{-2} - \epsilon \ln(x_{2n}) \quad (56)$$

is a solution of (55), and can be taken as invariant of the algebra.

(iii) $\mathfrak{r}_{2n+1}(\lambda_2^5, \dots, \lambda_2^{2n-1})$.

For the families the equation to be solved is

$$\widehat{Y}F := \sum_{k=2}^{2n-1} x_k \frac{\partial F}{\partial x_k} + 2x_{2n} \frac{\partial F}{\partial x_{2n}} = 0. \quad (57)$$

After some calculation it follows that

$$\widehat{Y}(I_1) = 2I_1 \quad \widehat{Y}(I_2) = 2I_2,$$

so that applying the same method as in (52), the invariant can be chosen as

$$J = \frac{I_2}{I_1}. \quad (58)$$

□

As expected, most of the solvable algebras have harmonics as invariants (see [13] for the invariants in the $\mathfrak{n}_{n,1}$ case). Only for special values of the parameters classical Casimir operators are obtained. It should be noted that no rational basis of invariants of $\mathfrak{r}_{2n+1}(2-n, \epsilon)$ can be obtained for $\epsilon \neq 0$.

Proposition 7. *The Lie algebra \mathfrak{r}_{2n+2} has no invariants for any $n \geq 3$.*

If $\{\omega_1, \dots, \omega_{2n}, \theta_1, \theta_2\}$ denotes a dual basis to $\{X_1, \dots, X_{2n}, Y_1, Y_2\}$, then the Maurer–Cartan equations have the form

$$\begin{aligned} d\omega_1 &= \omega_1 \wedge \theta_1 \\ d\omega_2 &= 2\omega_2 \wedge \theta_1 + \omega_2 \wedge \theta_2 \\ d\omega_k &= \omega_1 \wedge \omega_{k-1} + k\omega_k \wedge \theta_1 + \omega_k \wedge \theta_2, \quad 3 \leq k \leq 2n-1 \\ d\omega_{2n} &= \sum_{k=2}^n (-1)^k \omega_k \wedge \omega_{2n+1-k} + (2n+1)\omega_{2n} \wedge \theta_1 + 2\omega_{2n} \wedge \theta_2 \\ d\theta_1 &= d\theta_2 = 0. \end{aligned} \quad (59)$$

Taking the form $\xi = d\omega_1 + d\omega_{2n}$ and computing the n th wedge product we obtain

$$\bigwedge^n \xi = \pm(2n)n! \omega_1 \wedge \dots \wedge \omega_{2n} \wedge \theta_1 \wedge \theta_2 \neq 0, \quad (60)$$

and by formula (8) the Lie algebra has no invariants.

7. Geometric properties of solvable Lie algebras with Q_{2n} -radical

In view of the last proposition, which shows that the Lie algebra \mathfrak{r}_{2n+2} is endowed with an exact symplectic structure, it is natural to ask whether the other solvable Lie algebras with nilradical isomorphic to Q_{2n} and dimension $2n+1$ also have special geometrical properties. Specifically, we analyse the existence of linear contact forms on these algebras, i.e., 1-forms $\omega \in \mathfrak{r}_{2n+1}^*$ such that $\omega \wedge (\bigwedge^n d\omega) \neq 0$. This type of geometrical structure has been shown to be of interest for the analysis of differential equations and also for dynamical systems [22, 23].

Proposition 8. *Let $n \geq 3$. Any solvable Lie algebra \mathfrak{r} with nilradical isomorphic to Q_{2n} , with the exception of $\mathfrak{r}_{2n+1}(2-n, 0)$, is endowed with a linear contact form.*

Proof. Let $\{\omega_1, \dots, \omega_{2n}, \theta\}$ be a dual basis of $\{X_1, \dots, X_{2n}, Y\}$.

(i) The Maurer–Cartan equations of $\mathfrak{r}_{2n+1}(\lambda_2)$ are

$$\begin{aligned} d\omega_1 &= \omega_1 \wedge \theta \\ d\omega_2 &= \lambda_2 \omega_2 \wedge \theta \\ d\omega_k &= \omega_1 \wedge \omega_{k-1} + (k-2 + \lambda_2)\omega_k \wedge \theta, \quad 3 \leq k \leq 2n-1 \\ d\omega_{2n} &= \sum_{k=2}^n (-1)^k \omega_k \wedge \omega_{2n+1-k} + (2n-3 + 2\lambda_2)\omega_{2n} \wedge \theta \\ d\theta &= 0. \end{aligned} \quad (61)$$

Taking $\omega = \omega_1 + \omega_{2n}$, the exterior product gives

$$\omega \wedge \left(\bigwedge^n d\omega \right) = 2n(n-1)!(\lambda_2 + n - 2)\omega_1 \wedge \cdots \wedge \omega_{2n} \wedge \theta \neq 0. \tag{62}$$

(ii) The Maurer–Cartan equations of $\mathfrak{r}_{2n+1}(2-n, \varepsilon)$ are

$$\begin{aligned} d\omega_1 &= \omega_1 \wedge \theta \\ d\omega_2 &= \omega_2 \wedge \theta \\ d\omega_k &= \omega_1 \wedge \omega_{k-1} + (k-n)\omega_k \wedge \theta, \quad 3 \leq k \leq 2n-1 \\ d\omega_{2n} &= \sum_{k=2}^n (-1)^k \omega_k \wedge \omega_{2n+1-k} + \omega_{2n} \wedge \theta + \varepsilon\omega_1 \wedge \theta \\ d\theta &= 0. \end{aligned} \tag{63}$$

Taking again the 1-form $\omega = \omega_1 + \omega_{2n}$, we obtain that

$$\omega \wedge \left(\bigwedge^n d\omega \right) = \varepsilon n! \omega_1 \wedge \cdots \wedge \omega_{2n} \wedge \theta. \tag{64}$$

Thus ω is a contact form whenever $\varepsilon \neq 0$. It is not difficult to show that for $\varepsilon = 0$ there is no linear contact form.

(iii) For $\mathfrak{r}_{2n+1}(\lambda_2^5, \dots, \lambda_2^{2n-1})$, the Maurer–Cartan equations are quite complicated, due to the number of parameters λ_2^k . However, in order to find a contact form we can restrict ourselves to the following equations:

$$d\omega_1 = 0 \tag{65}$$

$$d\omega_{2n} = \sum_{k=2}^n (-1)^k \omega_k \wedge \omega_{2n+1-k} + 2\lambda_2 \omega_{2n} \wedge \theta. \tag{66}$$

Then the sum $\omega = \omega_1 + \omega_{2n}$ satisfies

$$\omega \wedge \left(\bigwedge^n d\omega \right) = 2n! \omega_1 \wedge \cdots \wedge \omega_{2n} \wedge \theta \neq 0, \tag{67}$$

and defines a contact form. □

In [23] it was shown that contact forms α on varieties imply the existence of a vector field X such that $\alpha(X) = 1$ and $X \lrcorner \alpha = 0$, called the dynamical system associated with α . Therefore for the previous solvable Lie algebras we can construct dynamical systems, which moreover have no singularities [23]. On the contrary, solvable Lie algebras having the nilpotent graded algebra $\mathfrak{n}_{n,1}$ as nilradical have a number of invariants which depends on the dimension, which implies that (in odd dimension) they cannot possess a contact form [24]. This loss of structure is due to the fact that the Heisenberg subalgebra of Q_{2n} spanned by $\{X_2, \dots, X_{2n}\}$ is contracted onto the maximal abelian subalgebra of $\mathfrak{n}_{n,1}$.

8. Quasi-classical subalgebras

In [25] the notion of quasi-classical Lie algebras was introduced to present abelian and semisimple gauge theories in a unified manner. Moreover, this approach allows us to construct gauge theories based on non-abelian and non-semisimple Lie algebras. In spite of the objection of having ghosts when the compactness condition² is not satisfied, quasi-classical algebras are

² That is, when the Lie algebra is reductive.

for the main difference between solvable Lie algebras with nilradical $\mathfrak{n}_{n,1}$ and Q_{2n} when realized as symmetry algebras of differential equations. As known, the Heisenberg algebra \mathfrak{h}_n of dimension $2n + 1$ can be faithfully realized in $k \geq (n + 1)$ -dimensional space by vector fields. Therefore any Lie algebra that contains \mathfrak{h}_n will need at least $(n + 1)$ -variables for any realization by vector fields. Since Q_{2n} contains \mathfrak{h}_{n-1} for any $n \geq 3$, the solvable Lie algebras with Q_{2n} -nilradical will appear as symmetries of partial differential equations in k -dimensional space, where $k \geq n$. As the algebra Q_{2n} is defined for $n \geq 3$, these solvable algebras do not describe dynamics of physical systems given by a system of ordinary differential equations. On the contrary, solvable Lie algebras with $\mathfrak{n}_{n,1}$ -nilradical allow planar realizations, and can therefore appear as symmetries of ordinary differential equations. Due to the simplicity of the invariants and structure for both cases, solvable Lie algebras with naturally graded nilradical of maximal nilindex are suitable candidates to analyse the problem of superposition formulae for nonlinear differential equations [28].

Finally, solvable Lie algebras analysed in this paper and in [13] are also of interest in the reduction of sourceless Yang–Mills equations by means of potentials with constant components. Since the nilpotent algebra $\mathfrak{n}_{n,1}$ contains an abelian ideal of codimension one [13], any rank one solvable Lie algebra associated with this nilradical will contain a codimension one solvable Lie subalgebra that only admits flat Yang–Mills potentials. By contrast, no solvable Lie subalgebra of solvable Lie algebras having Q_{2n} as nilradical has this property. Indeed, since these algebras always contain a Heisenberg subalgebra of dimension $2n - 1$, by a result of [29], any Lie algebra containing it necessarily admits a nonflat Yang–Mills potential. In conclusion, physically the solvable Lie algebras of proposition 5 exhibit a different behaviour from those of paper [13]. This fact moreover suggests that, even in the solvable case, group theoretical arguments based on graded Lie algebras are an adequate frame to analyse models for different physical phenomena.

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